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Parametric extensions of Shannon inequality and its reverse one in Hilbert space operators

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Abstract

We shall state the following parametric extensions of Shannon inequality and its reverse one in Hilbert space operators. *Let* $p \in [0,1]$ *and also let* $\{A_1, A_2, \ldots, A_n\}$ *and* $\{B_1, B_2, \ldots, B_n\}$ *be two sequences of strictly positive operators on a Hilbert space* H *such that* $\sum_{j=1}^{n} A_j \sharp_p B_j \leqslant I$. *Then*

$$
\begin{aligned}
\sum_{j=1}^{n} S_{p+1}(A_j|B_j) &\geqslant \left[\sum_{j=1}^{n}(A_j \natural_{p+1} B_j) + \left(I - \sum_{j=1}^{n} A_j \sharp_p B_j\right)\right] \\
&\quad \times \log\left[\sum_{j=1}^{n}(A_j \natural_{p+1} B_j) + \left(I - \sum_{j=1}^{n} A_j \sharp_p B_j\right)\right] \\
&\geqslant \log\left[\sum_{j=1}^{n}(A_j \natural_{p+1} B_j) + \left(I - \sum_{j=1}^{n} A_j \sharp_p B_j\right)\right] \\
&\geqslant \sum_{j=1}^{n} S_p(A_j|B_j) \\
&\geqslant -\log\left[\sum_{j=1}^{n}(A_j \natural_{p-1} B_j) + \left(I - \sum_{j=1}^{n} A_j \sharp_p B_j\right)\right]
\end{aligned}
$$

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$$\geqslant -\left[\sum_{j=1}^{n}(A_j \natural_{p-1} B_j) + \left(I - \sum_{j=1}^{n} A_j \sharp_p B_j\right)\right]$$

$$\times \log\left[\sum_{j=1}^{n}(A_j \natural_{p-1} B_j) + \left(I - \sum_{j=1}^{n} A_j \sharp_p B_j\right)\right]$$

$$\geqslant \sum_{j=1}^{n} S_{p-1}(A_j|B_j),$$

where $S_q(A|B) = A^{\frac{1}{2}}\left(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\right)^q\left(\log A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\right)A^{\frac{1}{2}}$ *for* $A > 0$, $B > 0$ *and any real number* q *and* $A \natural_q B = A^{\frac{1}{2}}\left(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\right)^q A^{\frac{1}{2}}$ *for* $A > 0$, $B > 0$ *and any real number* q. *In particular, if* $\sum_{j=1}^{n} A_j = \sum_{j=1}^{n} B_j = I$, *then*

$$\sum_{j=1}^{n} S_2(A_j|B_j) \geqslant \left[\sum_{j=1}^{n} B_j A_j^{-1} B_j\right] \log\left[\sum_{j=1}^{n} B_j A_j^{-1} B_j\right]$$

$$\geqslant \log\left[\sum_{j=1}^{n} B_j A_j^{-1} B_j\right] \geqslant \sum_{j=1}^{n} S_1(A_j|B_j) \geqslant 0$$

$$\geqslant \sum_{j=1}^{n} S(A_j|B_j) \geqslant -\log\left[\sum_{j=1}^{n} A_j B_j^{-1} A_j\right]$$

$$\geqslant -\left[\sum_{j=1}^{n} A_j B_j^{-1} A_j\right] \log\left[\sum_{j=1}^{n} A_j B_j^{-1} A_j\right]$$

$$\geqslant \sum_{j=1}^{n} S_{-1}(A_j|B_j),$$

where $S(A|B) = S_0(A|B) = A^{\frac{1}{2}}\left(\log A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\right)A^{\frac{1}{2}}$ *which is the relative operator entropy of* $A > 0$ *and* $B > 0$. Our results can be considered as parametric extensions of the following celebrated Shannon inequality [Ann. Math. Statistics 22 (1951) 79; Bull. Syst. Tech. J. 27 (1948) 379; Pitman Monographs and Surveys in Pure and Applied Mathematics 97, Addison Wesley Longman, 1998, p. 233] which is very useful and so famous in information theory. *Let* $\{a_1, a_2, \ldots, a_n\}$ *and* $\{b_1, b_2, \ldots, b_n\}$ *be two probability vectors. Then* $0 \geqslant \sum_{j=1}^{n} a_j \log b_j - \sum_{j=1}^{n} a_j \log a_j$ (see inequalities (2.4) of Corollary 2.4).

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Keywords: Shannon inequality; Reverse type Shannon inequality; Operator concave function

1. Introduction

First the Shannon inequality asserts [1]: Let $\{a_1, a_2, \dots, a_n\}$ and $\{b_1, b_2, \dots, b_n\}$ be two probability vectors. Then

$$0 \geqslant \sum_{j=1}^{n} a_j \log \frac{b_j}{a_j}. \tag{1.1}$$

We remark that $0 \geqslant \sum_{j=1}^{n} a_j \log \frac{b_j}{a_j}$ in (1.1) is equivalent to $D = \sum_{j=1}^{n} a_j \log \frac{a_j}{b_j} \geqslant 0$ which is the original number type Shannon inequality and this D is called "divergence" in [7,9].

In this paper we shall state parametric extensions of Shannon inequality and its reverse one in Hilbert space operators.

A bounded linear operator T on a Hilbert space H is said to be positive (denoted by $T \geqslant 0$) if $(Tx, x) \geqslant 0$ for all $x \in H$ and also an operator T is said to be strictly positive (denoted by $T > 0$) if T is invertible and positive.

Definition 1.1. $S_q(A|B)$ for $A > 0$, $B > 0$ and any real number q is defined by

$$S_q(A|B) = A^{\frac{1}{2}} \left(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}\right)^q \left(\log A^{\frac{-1}{2}} B A^{\frac{-1}{2}}\right) A^{\frac{1}{2}}.$$

We recall that $S_0(A|B) = A^{\frac{1}{2}}(\log A^{\frac{-1}{2}} B A^{\frac{-1}{2}})A^{\frac{1}{2}} = S(A|B)$ is the relative operator entropy in [2] and $S(A|I) = -A \log A$ is the usual operator entropy in [8].

Definition 1.2. $A \natural_q B$ for $A > 0$ and $B > 0$ and any real number q is defined by

$$A \natural_q B = A^{\frac{1}{2}} \left(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}\right)^q A^{\frac{1}{2}}$$

and $A \natural_p B$ for $p \in [0, 1]$ just coincides with $A \sharp_p B$ which is well known as p-power mean.

We remark that $S_1(A|B) = -S(B|A)$ and moreover $S_q(A|B) = -S_{1-q}(B|A)$ for any q.

Following after Definition 1.1, the original Shannon inequality can be expressed as follows:

$$0 \geqslant \sum_{j=1}^{n} a_j \log \frac{b_j}{a_j} = \sum_{j=1}^{n} a_j^{\frac{1}{2}} \left(\log a_j^{\frac{-1}{2}} b_j a_j^{\frac{-1}{2}}\right) a_j^{\frac{1}{2}} = \sum_{j=1}^{n} S(a_j|b_j).$$

Consequently $0 \geqslant \sum_{j=1}^{n} S(a_j|b_j)$ in the original Shannon inequality can be extended to $0 \geqslant \sum_{j=1}^{n} S(A_j|B_j)$ in operator version case (2.4) of Corollary 2.4, so that the form of (1.1) is convenient for operator type extension. We can summarize the following contrast:

The original Shannon inequality and its reverse one	*The operator version Shannon inequality and its reverse one*
$0 \geqslant \sum_{j=1}^{n} a_j \log \frac{b_j}{a_j} \geqslant -\log \sum_{j=1}^{n} \frac{a_j^2}{b_j}$	$0 \geqslant \sum_{j=1}^{n} S(A_j\|B_j) \geqslant -\log \sum_{j=1}^{n} A_j B_j^{-1} A_j$
for $a_j, b_j > 0$ *with* $1 = \sum_{j=1}^{n} a_j = \sum_{j=1}^{n} b_j$.	*for* $A_j, B_j > 0$ *with* $I = \sum_{j=1}^{n} A_j = \sum_{j=1}^{n} B_j$.

2. Parametric extensions of operator reverse type Shannon inequality derived from two operator concave functions $f_1(t) = \log t$ and $f_2(t) = -t \log t$

Firstly we shall state the following parametric extensions of Shannon inequality and its reverse one in Hilbert space operators derived from an operator concave function $f(t) = \log t$.

Theorem 2.1. *Let* $p \in [0, 1]$ *and also let* $\{A_1, A_2, \ldots, A_n\}$ *and* $\{B_1, B_2, \ldots, B_n\}$ *be two sequences of strictly positive operators on a Hilbert space* H *such that* $\sum_{j=1}^{n} A_j \sharp_p B_j \leqslant I$, *where* I *means the identity operator on* H. *Then*

$$\begin{aligned}
&\log\left[\sum_{j=1}^{n}(A_j \natural_{p+1} B_j) + t_0\left(I - \sum_{j=1}^{n} A_j \sharp_p B_j\right)\right] \\
&\quad -\log t_0\left(I - \sum_{j=1}^{n} A_j \sharp_p B_j\right) \\
&\qquad \geqslant \sum_{j=1}^{n} S_p(A_j|B_j) \\
&\qquad \geqslant -\log\left[\sum_{j=1}^{n}(A_j \natural_{p-1} B_j) + t_0\left(I - \sum_{j=1}^{n} A_j \sharp_p B_j\right)\right] \\
&\qquad\quad + \log t_0\left(I - \sum_{j=1}^{n} A_j \sharp_p B_j\right)
\end{aligned}$$

for fixed real number $t_0 > 0$, (2.1)

where $S_p(A|B)$ *is defined in Definition* 1.1 *and* $A \natural_q B$ *is defined in Definition* 1.2.

Secondly we shall state the following parametric extensions of Shannon inequality and its reverse one in Hilbert space operators derived from an operator concave function $f(t) = -t \log t$.

Theorem 2.2. *Let $p \in [0, 1]$ and also let $\{A_1, A_2, \ldots, A_n\}$ and $\{B_1, B_2, \ldots, B_n\}$ be two sequences of strictly positive operators on a Hilbert space H such that $\sum_{j=1}^{n} A_j \sharp_p B_j \leqslant I$, where I means the identity operator on H. Then*

$$\sum_{j=1}^{n} S_{p+1}(A_j|B_j) \geqslant \left[\sum_{j=1}^{n}(A_j \natural_{p+1} B_j) + t_0\left(I - \sum_{j=1}^{n} A_j \sharp_p B_j\right)\right] \times \log\left[\sum_{j=1}^{n}(A_j \natural_{p+1} B_j) + t_0\left(I - \sum_{j=1}^{n} A_j \sharp_p B_j\right)\right] - t_0 \log t_0 \left(I - \sum_{j=1}^{n} A_j \sharp_p B_j\right)$$

for fixed real number $t_0 > 0$ (2.2)

and

$$\sum_{j=1}^{n} S_{p-1}(A_j|B_j) \leqslant -\left[\sum_{j=1}^{n}(A_j \natural_{p-1} B_j) + t_0\left(I - \sum_{j=1}^{n} A_j \sharp_p B_j\right)\right] \times \log\left[\sum_{j=1}^{n}(A_j \natural_{p-1} B_j) + t_0\left(I - \sum_{j=1}^{n} A_j \sharp_p B_j\right)\right] + t_0 \log t_0 \left(I - \sum_{j=1}^{n} A_j \sharp_p B_j\right)$$

for fixed real number $t_0 > 0$, (2.2′)

where $S_q(A|B)$ is defined in Definition 1.1 *and $A \natural_q B$ is defined in Definition* 1.2.

We shall state the following result which can be shown by combining Theorem 2.1 with Theorem 2.2.

Corollary 2.3. *Let $p \in [0, 1]$ and also let $\{A_1, A_2, \ldots, A_n\}$ and $\{B_1, B_2, \ldots, B_n\}$ be two sequences of strictly positive operators on a Hilbert space H such that $\sum_{j=1}^{n} A_j \sharp_p B_j \leqslant I$, where I means the identity operator on H. Then*

$$\sum_{j=1}^{n} S_{p+1}(A_j|B_j) \geqslant \left[\sum_{j=1}^{n}(A_j \natural_{p+1} B_j) + \left(I - \sum_{j=1}^{n} A_j \sharp_p B_j\right)\right]$$

$$
\begin{aligned}
&\quad \times \log\left[\sum_{j=1}^{n}(A_j \natural_{p+1} B_j) + \left(I - \sum_{j=1}^{n} A_j \sharp_p B_j\right)\right] \\
&\geqslant \log\left[\sum_{j=1}^{n}(A_j \natural_{p+1} B_j) + \left(I - \sum_{j=1}^{n} A_j \sharp_p B_j\right)\right] \\
&\geqslant \sum_{j=1}^{n} S_p(A_j|B_j) \\
&\geqslant -\log\left[\sum_{j=1}^{n}(A_j \natural_{p-1} B_j) + \left(I - \sum_{j=1}^{n} A_j \sharp_p B_j\right)\right] \\
&\geqslant -\left[\sum_{j=1}^{n}(A_j \natural_{p-1} B_j) + \left(I - \sum_{j=1}^{n} A_j \sharp_p B_j\right)\right] \\
&\quad \times \log\left[\sum_{j=1}^{n}(A_j \natural_{p-1} B_j) + \left(I - \sum_{j=1}^{n} A_j \sharp_p B_j\right)\right] \\
&\geqslant \sum_{j=1}^{n} S_{p-1}(A_j|B_j), \qquad (2.3)
\end{aligned}
$$

where $S_q(A|B)$ *is defined in Definition* 1.1 *and* $A\natural_q B$ *is defined in Definition* 1.2.

Corollary 2.3 easily implies the following result which can be considered as operator version of Shannon inequality and its reverse one.

Corollary 2.4. *Let* $\{A_1, A_2, \ldots, A_n\}$ *and* $\{B_1, B_2, \ldots, B_n\}$ *be two sequences of strictly positive operators on a Hilbert space* H. *If* $\sum_{j=1}^{n} A_j = \sum_{j=1}^{n} B_j = I$, *then*

$$
\begin{aligned}
\sum_{j=1}^{n} S_2(A_j|B_j) &\geqslant \left[\sum_{j=1}^{n} B_j A_j^{-1} B_j\right] \log\left[\sum_{j=1}^{n} B_j A_j^{-1} B_j\right] \\
&\geqslant \log\left[\sum_{j=1}^{n} B_j A_j^{-1} B_j\right] \\
&\geqslant \sum_{j=1}^{n} S_1(A_j|B_j) \geqslant 0 \geqslant \sum_{j=1}^{n} S(A_j|B_j)
\end{aligned}
$$

$$
\begin{aligned}
&\geqslant -\log\left[\sum_{j=1}^{n} A_j B_j^{-1} A_j\right] \\
&\geqslant -\left[\sum_{j=1}^{n} A_j B_j^{-1} A_j\right]\log\left[\sum_{j=1}^{n} A_j B_j^{-1} A_j\right] \\
&\geqslant \sum_{j=1}^{n} S_{-1}(A_j|B_j). \qquad (2.4)
\end{aligned}
$$

Remark 2.1. We recall $S_q(A|B)$ for $A > 0$, $B > 0$ and any real number q as follows:

$$S_q(A|B) = A^{\frac{1}{2}}\left(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\right)^q\left(\log A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\right)A^{\frac{1}{2}}.$$

By an easy calculation we have

$$\frac{\mathrm{d}}{\mathrm{d}q}[S_q(A|B)] = A^{\frac{1}{2}}\left(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\right)^q\left[\log A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\right]^2A^{\frac{1}{2}} \geqslant 0,$$

so that $S_q(A|B)$ is an increasing function of q, and it is interesting to point out that the decreasing order of the positions of $\sum_{j=1}^{n} S_2(A_j|B_j)$, $\sum_{j=1}^{n} S_1(A_j|B_j)$, $\sum_{j=1}^{n} S(A_j|B_j)$, and $\sum_{j=1}^{n} S_{-1}(A_j|B_j)$ in (2.4) of Corollary 2.4 is quite reasonable since $\sum_{j=1}^{n} S(A_j|B_j) = \sum_{j=1}^{n} S_0(A_j|B_j)$.

3. Propositions needed to give proofs of the results in Section 2

By careful scrutinizing nice proofs in [5, Theorem 2.1] and [4, Theorem], we have the following parallel result to [5, Theorem 2.1].

Proposition 3.1. *If f is a continuous, real function on an interval J, the following conditions are equivalent*:

(i) *f is operator concave.*
(ii) *$f(C^*AC + t_0(I - C^*C)) \geqslant C^*f(A)C + f(t_0)(I - C^*C)$ for operator C with $\|C\| \leqslant 1$ and self-adjoint operator A with $\sigma(A) \subseteq J$ and for fixed real number $t_0 \in J$.*
(iii) *$f\big(\sum_{j=1}^{n} C_j^*A_jC_j + t_0\big(I - \sum_{j=1}^{n} C_j^*C_j\big)\big) \geqslant \sum_{j=1}^{n} C_j^*f(A_j)C_j + f(t_0)\big(I - \sum_{j=1}^{n} C_j^*C_j\big)$ for operators C_j with $\sum_{j=1}^{n} C_j^*C_j \leqslant I$ and self-adjoint operators A_j with $\sigma(A_j) \subseteq J$ for $j = 1, 2, \ldots, n$ and for fixed real number $t_0 \in J$.*

(iv) $f\left(\sum_{j=1}^{n} C_j^* A_j C_j\right) \geqslant \sum_{j=1}^{n} C_j^* f(A_j) C_j$ *for operators* C_j *with* $\sum_{j=1}^{n} C_j^* C_j = I$ *and self-adjoint operators* A_j *with* $\sigma(A_j) \subseteq J$ *for* $j = 1, 2, \ldots, n$, *where* $n \geqslant 2$.

(v) $f(PAP + t_0(I - P)) \geqslant Pf(A)P + f(t_0)(I - P)$ *for projection* P *and self-adjoint operator* A *with* $\sigma(A) \subseteq J$ *and for fixed real number* $t_0 \in J$.

Corollary 3.2. *If* f *is continuous operator concave function on the half open interval* $[0, \alpha)$ *to* $[0, \alpha)$ *with* $\alpha \leqslant \infty$, *then*

$$\begin{aligned} f\left(\sum_{j=1}^{n} C_j^* A_j C_j\right) &\geqslant \sum_{j=1}^{n} C_j^* f(A_j) C_j + f(0)\left(I - \sum_{j=1}^{n} C_j^* C_j\right) \\ &\geqslant \sum_{j=1}^{n} C_j^* f(A_j) C_j \end{aligned}$$

for operators C_j *with* $\sum_{j=1}^{n} C_j^* C_j \leqslant I$ *and self-adjoint operators* A_j *with* $\sigma(A_j) \subseteq [0, \alpha)$ *for* $j = 1, 2, \ldots, n$.

We recall the following obvious Proposition 3.3.

Proposition 3.3. *Let* $A > 0$ *and* $B > 0$. *Then* (i) $A \natural_{-1} B = AB^{-1}A$, (ii) $A \natural_2 B = BA^{-1}B$, (iii) $A \natural_0 B = A$, (iv) $A \natural_1 B = B$, *and* (v) $A \log A \geqslant \log A$ *for any* $A > 0$.

Remark 3.1. If (i$'$) f is continuous operator concave on J containing 0 and $f(0) \geqslant 0$, then the following (ii$'$) holds by (i) and (ii) of Proposition 5.1

$$\text{(ii}'\text{)} \quad f(C^*AC) \geqslant C^* f(A) C + f(0)(I - C^*C) \geqslant C^* f(A) C$$

for operator C with $\|C\| \leqslant 1$ and self-adjoint operator A with $\sigma(A) \subseteq J$ since $f(0) \geqslant 0$ and $I - C^*C \geqslant 0$.

As "f is continuous operator concave function and $f(0) \geqslant 0$" just essentially corresponds to "f is continuous operator convex function and $f(0) \leqslant 0$" in (i) of [5, Theorem 2.1], it turns out that Proposition 3.1 is essentially shown under an additional condition $f(0) \geqslant 0$ in [5, Theorem 2.1], briefly speaking, Proposition 3.1 with $f(0) \geqslant 0$ becomes Theorem 2.1 in [5].

Remark 3.2. It is shown in [6, Theorem 6] that if f is operator monotone function, (iv) of Proposition 3.1 holds. Also Corollary 3.2 implies that if f is an operator monotone function on the half open interval $[0, \alpha)$ to $[0, \alpha)$ with $\alpha \leqslant \infty$, then $f(\sum_{j=1}^{n} C_j^* A_j C_j) \geqslant \sum_{j=1}^{n} C_j^* f(A_j) C_j$ for operators C_j with $\sum_{j=1}^{n} C_j^* C_j \leqslant I$ and self-adjoint operators A_j with $\sigma(A_j) \subseteq [0, \alpha)$ for $j = 1, 2, \ldots, n$, which is shown in [6, Corollary 7], because f is operator concave on $[0, \alpha)$ to $[0, \alpha)$ with $\alpha \leqslant \infty$ if and only if f is operator monotone on $[0, \alpha)$ to $[0, \alpha)$ with $\alpha \leqslant \infty$.

Proof of Proposition 3.1. By carefull scrutinizing nice proofs in [5, Theorem 2.1] and [4, Theorem], we show the following implication. (i) $\longrightarrow$ (ii) $\longrightarrow$ (iii) $\longrightarrow$ (iv) $\longrightarrow$ (i) and (ii) $\longrightarrow$ (v) $\longrightarrow$ (i).

(i) $\longrightarrow$ (ii). Consider the operators $\mathbb{A}$, $\mathbb{U}$ and $\mathbb{V}$ on $H \oplus H$ defined by

$$\mathbb{A} = \begin{pmatrix} A & 0 \\ 0 & t_0 I \end{pmatrix}, \quad \mathbb{U} = \begin{pmatrix} C & (I - CC^*)^{\frac{1}{2}} \\ (I - C^*C)^{\frac{1}{2}} & -C^* \end{pmatrix},$$
$$\mathbb{V} = \begin{pmatrix} C & -(I - CC^*)^{\frac{1}{2}} \\ (I - C^*C)^{\frac{1}{2}} & C^* \end{pmatrix}.$$

It easily turns out that $\mathbb{U}$ and $\mathbb{V}$ are both unitary, $\mathbb{A}^* = \mathbb{A}$ and $\sigma(\mathbb{A}) = \sigma(A) \cup \sigma(t_0 I) \subseteq J$ and

$$\mathbb{U}^*\mathbb{A}\mathbb{U} = \begin{pmatrix} C^*AC + t_0(I - C^*C) & C^*A(I - CC^*)^{\frac{1}{2}} - t_0(I - C^*C)^{\frac{1}{2}}C^* \\ (I - CC^*)^{\frac{1}{2}}AC - t_0C(I - C^*C)^{\frac{1}{2}} & (I - CC^*)^{\frac{1}{2}}A(I - CC^*)^{\frac{1}{2}} + t_0CC^* \end{pmatrix}$$

and

$$\mathbb{V}^*\mathbb{A}\mathbb{V} = \begin{pmatrix} C^*AC + t_0(I - C^*C) & -C^*A(I - CC^*)^{\frac{1}{2}} + t_0(I - C^*C)^{\frac{1}{2}}C^* \\ -(I - CC^*)^{\frac{1}{2}}AC + t_0C(I - C^*C)^{\frac{1}{2}} & (I - CC^*)^{\frac{1}{2}}A(I - CC^*)^{\frac{1}{2}} + t_0CC^* \end{pmatrix}.$$

Since $\sigma(C^*AC + t_0(I - C^*C)) \subseteq J$, and $\sigma((I - CC^*)^{\frac{1}{2}}A(I - CC^*)^{\frac{1}{2}} + t_0CC^*) \subseteq J$, we obtain

$$\begin{aligned}
&\begin{pmatrix} f(C^*AC + t_0(I - C^*C)) & 0 \\ 0 & f\left((I - CC^*)^{\frac{1}{2}}A(I - CC^*)^{\frac{1}{2}} + t_0CC^*\right) \end{pmatrix} \\
&\quad = f\left(\begin{pmatrix} C^*AC + t_0(I - C^*C) & 0 \\ 0 & (I - CC^*)^{\frac{1}{2}}A(I - CC^*)^{\frac{1}{2}} + t_0CC^* \end{pmatrix}\right) \\
&\quad = f\left(\frac{1}{2}\mathbb{U}^*\mathbb{A}\mathbb{U} + \frac{1}{2}\mathbb{V}^*\mathbb{A}\mathbb{V}\right) \\
&\quad \geqslant \frac{1}{2}f(\mathbb{U}^*\mathbb{A}\mathbb{U}) + \frac{1}{2}f(\mathbb{V}^*\mathbb{A}\mathbb{V}) \quad \text{since } f \text{ is operator concave} \\
&\quad = \frac{1}{2}\mathbb{U}^*f(\mathbb{A})\mathbb{U} + \frac{1}{2}\mathbb{V}^*f(\mathbb{A})\mathbb{V} \quad \text{since } \mathbb{U} \text{ and } \mathbb{V} \text{ are both unitary} \\
&\quad = \begin{pmatrix} C^*f(A)C + f(t_0)(I - C^*C) & 0 \\ 0 & (I - CC^*)^{\frac{1}{2}}f(A)(I - CC^*)^{\frac{1}{2}} + f(t_0)CC^* \end{pmatrix}.
\end{aligned}$$

Comparing entries of the first operator matrix and the last one, in particular, so we have (ii) as follows:

$$f(C^*AC + t_0(I - C^*C)) \geqslant C^*f(A)C + f(t_0)(I - C^*C).$$

(ii) $\longrightarrow$ (iii). Consider the operators $\mathbb{X}$ and $\mathbb{Y}$ on $\mathbb{H} = \underbrace{H \oplus H \oplus \cdots \oplus H}_{n \text{ times}}$ defined by

$$\mathbb{X} = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & A_n \end{pmatrix}, \quad \mathbb{Y} = \begin{pmatrix} C_1 & 0 & \cdots & 0 \\ C_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C_n & 0 & \cdots & 0 \end{pmatrix}.$$

Then $\sigma(\mathbb{X}) = \bigcup_j \sigma(A_j) \subseteq J$, $\mathbb{X}^* = \mathbb{X}$ and $\mathbb{Y}^*\mathbb{Y} \leqslant \mathbb{I}$ since $\sum_{j=1}^n C_j^*C_j \leqslant I$ holds, where $\mathbb{I}$ means the identity of $\mathbb{H}$, and we recall

$$\begin{aligned} &\mathbb{Y}^*\mathbb{X}\mathbb{Y} + t_0(\mathbb{I} - \mathbb{Y}^*\mathbb{Y}) \\ &= \begin{pmatrix} \sum_{j=1}^n C_j^*A_jC_j + t_0\left(I - \sum_{j=1}^n C_j^*C_j\right) & 0 & \cdots & 0 \\ 0 & t_0I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_0I \end{pmatrix}. \end{aligned}$$

By (ii) we have the following since $\sum_{j=1}^n C_j^*A_jC_j + t_0(I - \sum_{j=1}^n C_j^*C_j) \subseteq J$, and $\sigma(t_0I) \subseteq J$,

$$f\left(\mathbb{Y}^*\mathbb{X}\mathbb{Y} + t_0(\mathbb{I} - \mathbb{Y}^*\mathbb{Y})\right) \geqslant \mathbb{Y}^*f(\mathbb{X})\mathbb{Y} + f(t_0)(\mathbb{I} - \mathbb{Y}^*\mathbb{Y})$$

that is,

$$\begin{aligned} &\begin{pmatrix} f\left(\sum_{j=1}^n C_j^*A_jC_j + t_0\left(I - \sum_{j=1}^n C_j^*C_j\right)\right) & 0 & \cdots & 0 \\ 0 & f(t_0)I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ddots & f(t_0)I \end{pmatrix} \\ &\geqslant \begin{pmatrix} \sum_{j=1}^n C_j^*f(A_j)C_j + f(t_0)\left(I - \sum_{j=1}^n C_j^*C_j\right) & 0 & \cdots & 0 \\ 0 & f(t_0)I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f(t_0)I \end{pmatrix} \end{aligned}$$

comparing entries, we have

$$f\left(\sum_{j=1}^n C_j^*A_jC_j + t_0\left(I - \sum_{j=1}^n C_j^*C_j\right)\right)$$

$$\geqslant \sum_{j=1}^{n} C_j^* f(A_j) C_j + f(t_0)\left(I - \sum_{j=1}^{n} C_j^* C_j\right)$$

so we have (iii).

(iii) $\longrightarrow$ (iv). We have only to take $\sum_{j=1}^{n} C_j^* C_j = I$ in (iii).

(iv) $\longrightarrow$ (i). Take C_1 and C_2 for real numbers c_1 and c_2 with $c_1^2 + c_2^2 = 1$ and take $C_j = 0$ for $j \geqslant 3$. Then $f(c_1^2 A_1 + c_2^2 A_2) \geqslant c_1^2 f(A_1) + c_2^2 f(A_2)$ holds by (iv), that is, f is concave, so we have (i).

(ii) $\longrightarrow$ (v). we have only to put $C = P$ in (ii).

(v) $\longrightarrow$ (i). Given self-adjoint operators A, B with $\sigma(A), \sigma(B) \subseteq J$ and $\lambda \in [0, 1]$, consider $\mathbb{X}$, $\mathbb{U}$ and $\mathbb{P}$ on $H \oplus H$ defined by

$$\mathbb{X} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad \mathbb{U} = \begin{pmatrix} \lambda^{\frac{1}{2}} I & -(1-\lambda)^{\frac{1}{2}} I \\ (1-\lambda)^{\frac{1}{2}} I & \lambda^{\frac{1}{2}} I \end{pmatrix}, \quad \mathbb{P} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $\sigma(\mathbb{X}) = \sigma(A) \cup \sigma(B) \subseteq J$, $\mathbb{X}^* = \mathbb{X}$, $\mathbb{U}$ is unitary and $\mathbb{P}$ is projection. By an easy calculation we have

$$\mathbb{P}\mathbb{U}^*\mathbb{X}\mathbb{U}\mathbb{P} + t_0(\mathbb{I} - \mathbb{P}) = \begin{pmatrix} \lambda A + (1-\lambda)B & 0 \\ 0 & t_0 I \end{pmatrix}. \tag{3.1}$$

$\mathbb{I}$ means the identity on $H \oplus H$. Since $\sigma(\lambda A + (1-\lambda)B) \subseteq J$, and $\sigma(t_0 I) \subseteq J$, we have

$$\begin{aligned} \begin{pmatrix} f(\lambda A + (1-\lambda)B) & 0 \\ 0 & f(t_0)I \end{pmatrix} &= f\begin{pmatrix} \lambda A + (1-\lambda)B & 0 \\ 0 & t_0 I \end{pmatrix} \\ &= f\left(\mathbb{P}\mathbb{U}^*\mathbb{X}\mathbb{U}\mathbb{P} + t_0(\mathbb{I} - \mathbb{P})\right) \quad \text{by (3.1)} \\ &\geqslant \mathbb{P} f(\mathbb{U}^*\mathbb{X}\mathbb{U})\mathbb{P} + f(t_0)(\mathbb{I} - \mathbb{P}) \quad \text{by (v)} \\ &= \mathbb{P}\mathbb{U}^* f(\mathbb{X})\mathbb{U}\mathbb{P} \\ &\quad + f(t_0)(\mathbb{I} - \mathbb{P}) \quad \text{since } \mathbb{U} \text{ is unitary} \\ &= \begin{pmatrix} \lambda f(A) + (1-\lambda) f(B) & 0 \\ 0 & f(t_0)I \end{pmatrix}, \end{aligned}$$

so that comparing entries, we have (i) as follows: $f(\lambda A + (I-\lambda)B) \geqslant \lambda f(A) + (1-\lambda) f(B)$.

Whence the proof of Proposition 3.1 is complete. □

Proof of Corollary 3.2. Obvious by (iii) of Proposition 3.1. □

Proof of Proposition 3.3. Obvious. □

4. Proofs of the results in Section 2

Proof of Theorem 2.1. First of all we state the following inequality (4.1) for any strictly positive operator $X_j > 0$ for $j = 1, 2, \ldots, n$ and for any fixed real number $t_0 > 0$ by (iii) of Proposition 3.1 since $\log t$ is operator concave function on $(0, \infty)$ and:

$$\log\left[\sum_{j=1}^{n} C_j^* X_j C_j + t_0\left(I - \sum_{j=1}^{n} C_j^* C_j\right)\right]$$
$$\geqslant \sum_{j=1}^{n} C_j^*(\log X_j)C_j + (\log t_0)\left(I - \sum_{j=1}^{n} C_j^* C_j\right) \tag{4.1}$$

for C_j for $j = 1, 2, \ldots, n$ with $\sum_{j=1}^{n} C_j^* C_j \leqslant I$. Put $X_j = \left(A_j^{\frac{-1}{2}} B_j A_j^{\frac{-1}{2}}\right)^q > 0$ for a real number q and $C_j = \left(A_j^{\frac{-1}{2}} B_j A_j^{\frac{-1}{2}}\right)^{\frac{p}{2}} A_j^{\frac{1}{2}}$ in (4.1). Then we recall $\sum_{j=1}^{n} C_j^* C_j = \sum_{j=1}^{n} A_j \natural_p B_j$.
Therefore (4.1) ensures the following inequality

$$\log\left[\sum_{j=1}^{n} A_j^{\frac{1}{2}}\left(A_j^{\frac{-1}{2}} B_j A_j^{\frac{-1}{2}}\right)^{p+q} A_j^{\frac{1}{2}} + t_0\left(I - \sum_{j=1}^{n} A_j \natural_p B_j\right)\right]$$
$$\geqslant \sum_{j=1}^{n} A_j^{\frac{1}{2}}\left(A_j^{\frac{-1}{2}} B_j A_j^{\frac{-1}{2}}\right)^{p}\left[\log\left(A_j^{\frac{-1}{2}} B_j A_j^{\frac{-1}{2}}\right)^{q}\right] A_j^{\frac{1}{2}}$$
$$+ (\log t_0)\left(I - \sum_{j=1}^{n} A_j \natural_p B_j\right)$$

for $A_j > 0$ and $B_j > 0$ for $j = 1, 2, \ldots, n$ with $\sum_{j=1}^{n} A_j \natural_p B_j \leqslant I$ and for any fixed real number $t_0 > 0$, we have the following inequality (4.2)

$$\log\left[\sum_{j=1}^{n}(A_j \natural_{p+q} B_j) + t_0\left(I - \sum_{j=1}^{n} A_j \natural_p B_j\right)\right]$$
$$\geqslant q\sum_{j=1}^{n} S_p(A_j|B_j) + (\log t_0)\left(I - \sum_{j=1}^{n} A_j \natural_p B_j\right). \tag{4.2}$$

Putting $q = 1$ and $q = -1$ in (4.2) respectively, we have

$$\begin{aligned}
&\log\left[\sum_{j=1}^{n}(A_j \natural_{p+1} B_j) + t_0\left(I - \sum_{j=1}^{n} A_j \sharp_p B_j\right)\right] \\
&\quad - \log t_0\left(I - \sum_{j=1}^{n} A_j \sharp_p B_j\right) \\
&\quad \geqslant \sum_{j=1}^{n} S_p(A_j|B_j) \\
&\quad \geqslant -\log\left[\sum_{j=1}^{n}(A_j \natural_{p-1} B_j) + t_0\left(I - \sum_{j=1}^{n} A_j \sharp_p B_j\right)\right] \\
&\qquad + \log t_0\left(I - \sum_{j=1}^{n} A_j \sharp_p B_j\right) \\
&\quad \text{for fixed real number } t_0 > 0.
\end{aligned} \tag{2.1}$$

Whence the proof of Theorem 2.1 is complete. □

Proof of Theorem 2.2. We recall (iii) of Proposition 3.1, that is, if f is a continuous, real operator concave function on an interval J, then

$$\begin{aligned}
&f\left(\sum_{j=1}^{n} C_j^* X_j C_j + t_0\left(I - \sum_{j=1}^{n} C_j^* C_j\right)\right) \\
&\quad \geqslant \sum_{j=1}^{n} C_j^* f(X_j) C_j + f(t_0)\left(I - \sum_{j=1}^{n} C_j^* C_j\right)
\end{aligned} \tag{4.3}$$

for operators C_j with $\sum_{j=1}^{n} C_j^* C_j \leqslant I$ and self-adjoint operators X_j with $\sigma(X_j) \subseteq J$ for $j = 1, 2, \ldots, n$ and for fixed real number $t_0 \in J$.

We put $f(t) = -t \log t$ in (4.3) since $-t \log t$ is an operator concave function on $(0, \infty)$, and put $X_j = \left(A_j^{\frac{-1}{2}} B_j A_j^{\frac{-1}{2}}\right)^q > 0$ for a real number q and $C_j = \left(A_j^{\frac{-1}{2}} B_j A_j^{\frac{-1}{2}}\right)^{\frac{p}{2}} A_j^{\frac{1}{2}}$ in (4.3). Then we recall $\sum_{j=1}^{n} C_j^* C_j = \sum_{j=1}^{n} A_j \sharp_p B_j$. Then (4.3) ensures the following inequality

$$\sum_{j=1}^{n} A_j^{\frac{1}{2}}\left(A_j^{\frac{-1}{2}} B_j A_j^{\frac{-1}{2}}\right)^{p+q}\left[\log\left(A_j^{\frac{-1}{2}} B_j A_j^{\frac{-1}{2}}\right)^q\right] A_j^{\frac{1}{2}}$$

$$
\begin{aligned}
&+ t_0 \log t_0 \left(I - \sum_{j=1}^{n} A_j \sharp_p B_j \right) \\
&\geqslant \left[\sum_{j=1}^{n} A_j^{\frac{1}{2}} \left(A_j^{\frac{-1}{2}} B_j A_j^{\frac{-1}{2}} \right)^{p+q} A_j^{\frac{1}{2}} + t_0 \left(I - \sum_{j=1}^{n} A_j \sharp_p B_j \right) \right] \\
&\quad \times \log \left[\sum_{j=1}^{n} A_j^{\frac{1}{2}} \left(A_j^{\frac{-1}{2}} B_j A_j^{\frac{-1}{2}} \right)^{p+q} A_j^{\frac{1}{2}} + t_0 \left(I - \sum_{j=1}^{n} A_j \sharp_p B_j \right) \right] \qquad (4.4)
\end{aligned}
$$

for $A_j > 0$ and $B_j > 0$ for $j = 1, 2, \ldots, n$ with $\sum_{j=1}^{n} A_j \sharp_p B_j \leqslant I$ and for any fixed real number $t_0 > 0$, we have the following inequality

$$
\begin{aligned}
&q \sum_{j=1}^{n} S_{p+q}(A_j | B_j) + t_0 \log t_0 \left(I - \sum_{j=1}^{n} A_j \sharp_p B_j \right) \\
&\geqslant \left[\sum_{j=1}^{n} A_j \sharp_{p+q} B_j + t_0 \left(I - \sum_{j=1}^{n} A_j \sharp_p B_j \right) \right] \\
&\quad \times \log \left[\sum_{j=1}^{n} A_j \sharp_{p+q} B_j + t_0 \left(I - \sum_{j=1}^{n} A_j \sharp_p B_j \right) \right] \qquad (4.5)
\end{aligned}
$$

for $A_j > 0$ and $B_j > 0$ for $j = 1, 2, \ldots, n$ with $\sum_{j=1}^{n} A_j \sharp_p B_j \leqslant I$ and for any fixed real number $t_0 > 0$. Putting $q = 1$ and $q = -1$ in (4.5) respectively, we have

$$
\begin{aligned}
\sum_{j=1}^{n} S_{p+1}(A_j | B_j) \geqslant &\left[\sum_{j=1}^{n} (A_j \natural_{p+1} B_j) + t_0 \left(I - \sum_{j=1}^{n} A_j \sharp_p B_j \right) \right] \\
&\times \log \left[\sum_{j=1}^{n} (A_j \natural_{p+1} B_j) + t_0 \left(I - \sum_{j=1}^{n} A_j \sharp_p B_j \right) \right] \\
&- t_0 \log t_0 \left(I - \sum_{j=1}^{n} A_j \sharp_p B_j \right) \\
&\text{for fixed real number } t_0 > 0 \qquad (2.2)
\end{aligned}
$$

and

$$
\sum_{j=1}^{n} S_{p-1}(A_j | B_j) \leqslant - \left[\sum_{j=1}^{n} (A_j \natural_{p-1} B_j) + t_0 \left(I - \sum_{j=1}^{n} A_j \sharp_p B_j \right) \right]
$$

$$\times \log \left[\sum_{j=1}^{n} (A_j \natural_{p-1} B_j) + t_0 \left(I - \sum_{j=1}^{n} A_j \sharp_p B_j \right) \right]$$
$$+ t_0 \log t_0 \left(I - \sum_{j=1}^{n} A_j \sharp_p B_j \right)$$

for fixed real number $t_0 > 0$. (2.2′)

Whence the proof of Theorem 2.2 is complete. □

Proof of Corollary 2.3. Put $t_0 = 1$ in Theorem 2.1 and Theorem 2.2 and (2.3) follows by (2.1), (2.2) and (2.2′) by using (v) $A \log A \geqslant \log A$ for any $A > 0$ in Proposition 3.3. □

Proof of Corollary 2.4. Put $p = 1$ in Corollary 2.3. Then $\sum_{j=1}^{n} A_j \sharp_1 B_j = \sum_{j=1}^{n} B_j$ by (iv) of Proposition 3.3. If $\sum_{j=1}^{n} A_j = \sum_{j=1}^{n} B_j = I$, then (2.3) of Corollary 2.3 implies

$$\begin{aligned}
\sum_{j=1}^{n} S_2(A_j|B_j) &\geqslant \left[\sum_{j=1}^{n} B_j A_j^{-1} B_j \right] \log \left[\sum_{j=1}^{n} B_j A_j^{-1} B_j \right] \\
&\geqslant \log \left[\sum_{j=1}^{n} B_j A_j^{-1} B_j \right] \\
&\geqslant \sum_{j=1}^{n} S_1(A_j|B_j) \geqslant -\log I \geqslant \sum_{j=1}^{n} S(A_j|B_j). \qquad (4.6)
\end{aligned}$$

Put $p = 0$ in Corollary 2.3. Then $\sum_{j=1}^{n} A_j \sharp_0 B_j = \sum_{j=1}^{n} A_j$ by (iii) of Proposition 3.3. If $\sum_{j=1}^{n} A_j = \sum_{j=1}^{n} B_j = I$, then (2.3) of Corollary 2.3 implies

$$\begin{aligned}
\sum_{j=1}^{n} S_1(A_j|B_j) &\geqslant \log I \geqslant \sum_{j=1}^{n} S(A_j|B_j) \\
&\geqslant -\log \left[\sum_{j=1}^{n} A_j B_j^{-1} A_j \right] \\
&\geqslant -\log \left[\sum_{j=1}^{n} A_j B_j^{-1} A_j \right] \left[\sum_{j=1}^{n} A_j B_j^{-1} A_j \right] \\
&\geqslant \sum_{j=1}^{n} S_{-1}(A_j|B_j). \qquad (4.7)
\end{aligned}$$

Whence we have (2.4) by combining (4.6) with (4.7). □

5. Precise estimation

Lemma 5.1. *If* $A > 0$ *and* $B > 0$, *then*

$$BA^{-1}B - B \geqslant S_1(A|B) \geqslant B - A \geqslant S(A|B) \geqslant A - AB^{-1}A. \tag{5.1}$$

Proof. We remark that the third inequality and fourth one in (5.1) are shown in [3]. We state a simple proof of (5.1). We recall the following well known inequality

$$Y - I \geqslant \log Y \geqslant I - Y^{-1} \quad \text{for } Y > 0 \tag{5.2}$$

and multiplying Y each terms in (5.2) and combining it with (5.2) itself, we have

$$Y^2 - Y \geqslant Y \log Y \geqslant Y - I \geqslant \log Y \geqslant I - Y^{-1} \quad \text{for } Y > 0. \tag{5.3}$$

Put $Y = A^{\frac{-1}{2}} B A^{\frac{-1}{2}}$ in (5.3) and multiplying $A^{\frac{1}{2}}$ on both sides, then we have (5.1). □

Remark 5.1. Lemma 5.1 easily implies the following result:

Let $\{A_1, A_2, \ldots, A_n\}$ and $\{B_1, B_2, \ldots, B_n\}$ be two sequences of strictly positive operators on a Hilbert space H such that $\sum_{j=1}^n A_j = \sum_{j=1}^n B_j = I$. Then

$$\left[\sum_{j=1}^n B_j A_j^{-1} B_j\right] - I \geqslant \sum_{j=1}^n S_1(A_j|B_j) \geqslant 0$$
$$\geqslant \sum_{j=1}^n S(A_j|B_j) \geqslant I - \left[\sum_{j=1}^n A_j B_j^{-1} A_j\right]. \tag{5.4}$$

We recall the following inequality by (2.4) in Corollary 2.4 under the same hypothesis $\sum_{j=1}^n A_j = \sum_{j=1}^n B_j = I$

$$\log\left[\sum_{j=1}^n B_j A_j^{-1} B_j\right] \geqslant \sum_{j=1}^n S_1(A_j|B_j) \geqslant 0$$
$$\geqslant \sum_{j=1}^n S(A_j|B_j) \geqslant -\log\left[\sum_{j=1}^n A_j B_j^{-1} A_j\right]. \tag{5.5}$$

Comparing (5.5) with (5.4), since $Y - I \geqslant \log Y$ for any $Y > 0$ it turns out that

$$\left[\sum_{j=1}^n B_j A_j^{-1} B_j\right] - I \geqslant \log\left[\sum_{j=1}^n B_j A_j^{-1} B_j\right]$$

and also

$$-\log\left[\sum_{j=1}^n A_j B_j^{-1} A_j\right] \geqslant I - \left[\sum_{j=1}^n A_j B_j^{-1} A_j\right],$$

that is, we could obtain more precise estimation (5.5) than (5.4) thanks to Proposition 3.1 which is characterization of operator concave functions.

Addendum

After we have written this manuscript, we know that quite similar results to Proposition 5.1 are shown in the recent paper of Hansen and Pedersem [10].

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